Online Resit Exam — Analysis (WBMA012-05)

Friday 16 April 2021, 18.30h–21.30h CEST (plus 30 minutes for uploading)

University of Groningen

Instructions

1. All answers need to be accompanied with an explanation or a calculation: only answering "yes", "no", or "42" is not sufficient.
2. If p is the number of marks then the exam grade is $G = 1 + p/10$.
3. Write both your name and student number on the answer sheets!
4. This exam comes in two versions. Both versions consist of six problems of equal difficulty.

   Make version 1 if your student number is odd.

   Make version 2 if your student number is even.

   For example, if your student number is 1277456, which is even, then you have to make version 2.
5. Save your work as a single PDF file and submit it via this dedicated Nestor page.

Version 1: odd student numbers only!

Problem 1 (5 + 5 + 5 = 15 points)

Assume that $A \subset \mathbb{R}$ is nonempty. The distance between a point $x \in \mathbb{R}$ and the set A, denoted by $d(x, A)$, is defined as

$$d(x, A) = \inf\{|x - a| \,:\, a \in A\}.$$

(a) Explain why the infimum above exists.

(b) Compute $d(x, A)$ for $x = 2$ and $A = (0, 1)$. Motivate your answer.

(c) Compute $d(x, A)$ for $x = \sqrt{2}$ and $A = \mathbb{Q}$. Motivate your answer.

Problem 2 (4 + 8 + 3 = 15 points)

Consider the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$, where $a_k = \begin{cases} 1/2^{k-1} & \text{if } k \text{ is odd,} \\ 1/k & \text{if } k \text{ is even.} \end{cases}$

(a) Show that the sum of the first $2n$ terms of the series is given by

$$s_{2n} = \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

(b) Prove that the series $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ diverges.

(c) Why does part (b) not contradict the Alternating Series Test?

Problem 3 (5 + 7 + 3 = 15 points)

Assume that $K \subset \mathbb{R}$ is compact. In addition, assume that $A \subset K$ and that A has infinitely many elements.

Prove the following statements:

(a) If $x \in K$ is *not* a limit point of A, then there exists $\epsilon_x > 0$ such that $V_{\epsilon_x}(x)$ contains at most one point of A.

(b) If no point of K is a limit point of A, then K has an open cover without a finite subcover.

(c) The set A has a limit point which is contained in K.

Problem 4 (5 + 5 + 5 = 15 points)

(a) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree n and assume that

$$\sum_{k=0}^{n} \frac{a_k}{k+1} = 0.$$

Use the Mean Value Theorem to show that $p(c) = 0$ for some $c \in (0,1)$.

(b) Show that $q(x) = 20x^4 - 12x^3 + 6x^2 - 14x + 4$ has a root in the interval $(0,1)$.

(c) Show that the derivative of q also has a root in the interval $(0,1)$.
Hint: compute $q(0)$ and $q(1)$.

Problem 5 (3 + 6 + 6 = 15 points)

Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, and consider the functions

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = f(x + 1/n).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to f.

(b) If $f(x) = \sin(x)$, then (f_n) converges uniformly on $\mathbb{R}$.

(c) If $f(x) = 5x^2 + 3$, then (f_n) does *not* converge uniformly on $\mathbb{R}$.

Problem 6 (3 + 6 + 3 + 3 = 15 points)

Consider the functions $f, g : [0, \pi/2] \to \mathbb{R}$ given by

$$f(x) = \cos(x) \quad \text{and} \quad g(x) = \begin{cases} \cos(x) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let P be any partition of the interval $[0, \pi/2]$.

Prove the following statements:

(a) $L(g, P) = 0$.

(b) $U(g, P) = U(f, P)$.

(c) $U(f, P) \geq 1$.

(d) The function g is not integrable on $[0, \pi/2]$.

End of version 1 (90 points)

Version 2: even student numbers only!

Problem 1 (5 + 5 + 5 = 15 points)

Assume that $A \subset \mathbb{R}$ is nonempty. The distance between a point $x \in \mathbb{R}$ and the set A, denoted by $d(x, A)$, is defined as

$$d(x, A) = \inf\{|x - a| \,:\, a \in A\}.$$

(a) Explain why the infimum above exists.

(b) Compute $d(x, A)$ for $x = 3$ and $A = (0, 1)$. Motivate your answer.

(c) Compute $d(x, A)$ for $x = \sqrt{3}$ and $A = \mathbb{Q}$. Motivate your answer.

Problem 2 (4 + 8 + 3 = 15 points)

Consider the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$, where $a_k = \begin{cases} 1/3^{k-1} & \text{if } k \text{ is odd,} \\ 1/k & \text{if } k \text{ is even.} \end{cases}$

(a) Show that the sum of the first $2n$ terms of the series is given by

$$s_{2n} = \left(1 + \frac{1}{9} + \frac{1}{9^2} + \cdots + \frac{1}{9^{n-1}}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

(b) Prove that the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ diverges.

(c) Why does part (b) not contradict the Alternating Series Test?

Problem 3 (5 + 7 + 3 = 15 points)

Assume that $K \subset \mathbb{R}$ is compact. In addition, assume that $A \subset K$ and that A has infinitely many elements.

Prove the following statements:

(a) If $x \in K$ is *not* a limit point of A, then there exists $\epsilon_x > 0$ such that $V_{\epsilon_x}(x)$ contains at most one point of A.

(b) If no point of K is a limit point of A, then K has an open cover without a finite subcover.

(c) The set A has a limit point which is contained in K.

Problem 4 (5 + 5 + 5 = 15 points)

(a) Let $p(x) = \sum_{k=0}^{n} a_k x^k$ be a polynomial of degree n and assume that

$$\sum_{k=0}^{n} \frac{a_k}{k+1} = 0.$$

Use the Mean Value Theorem to show that $p(c) = 0$ for some $c \in (0,1)$.

(b) Show that $q(x) = 10x^4 - 6x^3 + 3x^2 - 7x + 2$ has a root in the interval $(0,1)$.

(c) Show that the derivative of q also has a root in the interval $(0,1)$.
Hint: compute $q(0)$ and $q(1)$.

Problem 5 (3 + 6 + 6 = 15 points)

Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous, and consider the functions

$$f_n : \mathbb{R} \to \mathbb{R}, \qquad f_n(x) = f(x + 1/n).$$

Prove the following statements:

(a) The sequence (f_n) converges pointwise to f.

(b) If $f(x) = \cos(x)$, then (f_n) converges uniformly on $\mathbb{R}$.

(c) If $f(x) = 3x^2 + 5$, then (f_n) does *not* converge uniformly on $\mathbb{R}$.

Problem 6 (3 + 6 + 3 + 3 = 15 points)

Consider the functions $f, g : [0, \pi/2] \to \mathbb{R}$ given by

$$f(x) = \sin(x) \quad \text{and} \quad g(x) = \begin{cases} \sin(x) & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Let P be any partition of the interval $[0, \pi/2]$.

Prove the following statements:

(a) $L(g, P) = 0$.

(b) $U(g, P) = U(f, P)$.

(c) $U(f, P) \geq 1$.

(d) The function g is not integrable on $[0, \pi/2]$.

End of version 2 (90 points)

Solution of problem 1 (5 + 5 + 5 = 15 points)

(a) *Method 1 (slightly incomplete, but acceptable).* We have that $0 \le |x-a|$ for all $a \in A$, which means that the set $\{|x-a| \,:\, a \in A\}$ is bounded below. By the Axiom of Completeness any set that is bounded below has a greatest lower bound.

(This argument is slightly incomplete since the Axiom of Completeness only asserts the existence of least upper bounds. However, the existence of greatest lower bounds can be deduced from this.)
(5 points)

Method 2 (more complete). We have that $-|x-a| \le 0$ for all $a \in A$, which means that the set $\{-|x-a| \,:\, a \in A\}$ is bounded above. By the Axiom of Completeness any set that is bounded above also has a greatest lower bound. So $s = \sup\{-|x-a| \,:\, a \in A\}$ exists, and $-s$ is the greatest lower bound of $\{|x-a| \,:\, a \in A\}$.
(5 points)

(b) *Version 1.* If $x = 2$, then $|x-a| = |2-a| = 2-a > 1$ for all $a \in A$. This means that the set $\{|x-a| \,:\, a \in A\}$ is bounded below by $\ell = 1$.
(2 points)

Method 1. To prove that $\ell = 1$ is the greatest lower bound, we need to show that for each $\epsilon > 0$ there exists $a \in A$ such that $|x-a| < 1+\epsilon$. Taking $a \in A$ such that $a > 1-\epsilon$ implies that $-a < \epsilon - 1$ and thus

$$|x-a| = |2-a| = 2-a < 2+(\epsilon-1) = 1+\epsilon.$$

We conclude that $d(x, A) = 1$.
(3 points)

Method 2. Assume that ℓ' is any lower bound for the set $\{|x-a| \,:\, a \in A\}$, which means that $\ell' \le |x-a| = 2-a$ for all $a \in A$. In particular, we have that

$$\ell' \le 2 - \frac{n}{n+1}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ and using the Order Limit Theorem gives $\ell' \le 1$. This shows that $\ell = 1$ is the greatest lower bound and we conclude that $d(x, A) = 1$.
(3 points)

Version 2. If $x = 3$, then $|x-a| = |3-a| = 3-a > 2$ for all $a \in A$. This means that the set $\{|x-a| \,:\, a \in A\}$ is bounded below by $\ell = 2$.
(2 points)

Method 1. To prove that $\ell = 2$ is the greatest lower bound, we need to show that for each $\epsilon > 0$ there exists $a \in A$ such that $|x-a| < 2+\epsilon$. Taking $a \in A$ such that $a > 1-\epsilon$ implies that $-a < \epsilon - 1$ and thus

$$|x-a| = |3-a| = 3-a < 3+(\epsilon-1) = 2+\epsilon.$$

We conclude that $d(x, A) = 2$.
(3 points)

Method 2. Assume that ℓ' is any lower bound for the set $\{|x-a| \,:\, a \in A\}$, which means that $\ell' \le |x-a| = 3-a$ for all $a \in A$. In particular, we have that

$$\ell' \le 3 - \frac{n}{n+1}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$ and using the Order Limit Theorem gives $\ell' \leq 2$. This shows that $\ell = 1$ is the greatest lower bound and we conclude that $d(x, A) = 2$.
(3 points)

(c) *Version 1.* First observe that $|\sqrt{2} - r| \geq 0$ for all $r \in \mathbb{Q}$, which means that the set $\{|\sqrt{2} - r| : r \in \mathbb{Q}\}$ is bounded below by $\ell = 0$.
(2 points)

To prove that $\ell = 0$ is the greatest lower bound, we need to show that for each $\epsilon > 0$ there exists $r \in \mathbb{Q}$ such that $|\sqrt{2} - r| < \epsilon$. This is possible since the rational numbers are dense in the real numbers: for all $a, b \in \mathbb{R}$ with $a < b$ there exists $r \in \mathbb{Q}$ such that $a < r < b$. In particular, this holds for $a = \sqrt{2}$ and $b = \sqrt{2} + \epsilon$. We conclude that $d(\sqrt{2}, \mathbb{Q}) = 0$.
(3 points)

Version 2. Simply replace $\sqrt{2}$ by $\sqrt{3}$ in the argument above.

Solution of problem 2 (4 + 8 + 3 = 15 points)

(a) The first $2n$ terms of the series are given by

$$\begin{aligned}
s_{2n} &= \sum_{k=1}^{2n} (-1)^{k+1} a_k \\
&= a_1 - a_2 + a_3 - a_4 + \cdots + a_{2n-1} - a_{2n} \\
&= (a_1 + a_3 + \cdots + a_{2n-1}) - (a_2 + a_4 + \cdots + a_{2n}) \\
&= \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2n-2}}\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n}\right) \\
&= \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}}\right) - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right).
\end{aligned}$$

(4 points)

(b) Write $s_{2n} = g_n - \frac{1}{2}h_n$ where

$$g_n = 1 + \frac{1}{4} + \frac{1}{4^2} + \cdots + \frac{1}{4^{n-1}} \quad \text{and} \quad h_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

The sequence (g_n) converges since the terms of this sequence are the partial sums of a convergent geometric series.
(2 points)

The sequence (h_n) does not converge since the terms of this sequence are the partial sums of the harmonic series.
(2 points)

If the sequence (s_{2n}) would converge, then by the Algebraic Limit Theorem the sequence (h_n) would also converge as $h_n = 2(g_n - s_{2n})$. This is a contradiction. Therefore, the sequence (s_{2n}) does not converge.
(2 points)

If the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ would converge, then the sequence of partial sums (s_n) would converge. In particular, the subsequence (s_{2n}) would converge, but we know this is not the case. We conclude that the series $\sum_{k=1}^{\infty}(-1)^{k+1}a_k$ diverges.
(2 points)

(c) The sequence (a_k) is not decreasing. Therefore, the Alternating Series Test does not apply to this example even though we have an alternating series and $\lim a_k = 0$.
(3 points)

Solution of problem 3 (5 + 7 + 3 = 15 points)

(a) If $x \in K$ is a limit point of A, then by definition it follows that for all $\epsilon > 0$ the set $V_\epsilon(x)$ intersects the set A in some point different from x.
(2 points)

So if $x \in K$ is *not* a limit point of A, then there exists $\epsilon_x > 0$ such that either $V_{\epsilon_x}(x) \cap A = \{x\}$ (in case $x \in A$) or $V_{\epsilon_x}(x) \cap A = \varnothing$ (in case $x \notin A$). Therefore, the set $V_{\epsilon_x}(x) \cap A$ contains at most one point of A.
(3 points)

(b) If no point $x \in K$ is a limit point of A, then by part (a) there exist numbers $\epsilon_x > 0$ such that $V_{\epsilon_x}(x) \cap A$ contains at most one point of A. The sets $V_{\epsilon_x}(x)$ form an open cover for K. Indeed, since $x \in V_{\epsilon_x}(x)$ we have that

$$K \subset \bigcup_{x \in K} V_{\epsilon_x}(x).$$

(2 points)

We claim that this open cover does not have a finite subcover. Indeed, assume that

$$K \subset V_{\epsilon_{x_1}}(x_1) \cup \cdots \cup V_{\epsilon_{x_n}}(x_n).$$

(2 points)

If we intersect both sides of this inclusion by the set A we obtain

$$A \subset (V_{\epsilon_{x_1}}(x_1) \cap A) \cup \cdots \cup (V_{\epsilon_{x_n}}(x_n) \cap A).$$

By part (a) the right hand contains only finitely many points. But this contradicts the assumption that A has infinitely many elements. We conclude that K has an open cover without a finite subcover.
(3 points)

(c) Part (b) contradicts the assumption that K is compact since compact sets have the property that any open cover has a finite subcover. Therefore, A must have a limit point which is contained in K.
(3 points)

Solution of problem 4 (5 + 5 + 5 = 15 points)

(a) Consider an antiderivative of p given by

$$P(x) = r + \sum_{k=0}^{n} \frac{a_k}{k+1} x^{k+1},$$

where r is an arbitrary constant. Applying the Mean Value Theorem to P on the interval $[0,1]$ implies that there exists $c \in (0,1)$ such that

$$P'(c) = \frac{P(1) - P(0)}{1 - 0}.$$

This gives

$$p(c) = \sum_{k=0}^{n} \frac{a_k}{k+1} = 0,$$

where the last equality follows from the assumption.
(5 points)

(b) *Version 1.* For $q(x) = 20x^4 - 12x^3 + 6x^2 - 14x + 4$ we have that

$$\sum_{k=0}^{n} \frac{a_k}{k+1} = \frac{4}{1} - \frac{14}{2} + \frac{6}{3} - \frac{12}{4} + \frac{20}{5} = 4 - 7 + 2 - 3 + 4 = 0.$$

Applying part (a) gives the existence of $c \in (0,1)$ for which $q(c) = 0$.
(5 points)

Version 2. For $q(x) = 10x^4 - 6x^3 + 3x^2 - 7x + 2$ we have that

$$\sum_{k=0}^{n} \frac{a_k}{k+1} = \frac{2}{1} - \frac{7}{2} + \frac{3}{3} - \frac{6}{4} + \frac{10}{5} = 2 - \left(3 + \frac{1}{2}\right) + 1 - \left(1 + \frac{1}{2}\right) + 2 = 0.$$

Applying part (a) gives the existence of $c \in (0,1)$ for which $q(c) = 0$.
(5 points)

(c) *Version 1.* For $q(x) = 20x^4 - 12x^3 + 6x^2 - 14x + 4$ we have that $q(0) = 4$. We also have that

$$q(1) = 20 - 12 + 6 - 14 + 4 = 4.$$

Since $q(0) = q(1)$ it follows by either Rolle's Theorem or the Mean Value Theorem that there exists $c \in (0,1)$ such that $q'(c) = 0$.
(5 points)

Version 2. For $q(x) = 10x^4 - 6x^3 + 3x^2 - 7x + 2$ we have that $q(0) = 2$. We also have that

$$q(1) = 10 - 6 + 3 - 7 + 2 = 2.$$

Since $q(0) = q(1)$ it follows by either Rolle's Theorem or the Mean Value Theorem that there exists $c \in (0,1)$ such that $q'(c) = 0$.
(5 points)

Solution of problem 5 (3 + 6 + 6 = 15 points)

(a) *Method 1.* Let $x \in \mathbb{R}$ be fixed. Since f is continuous at x we have for all sequences (x_n) with $\lim x_n = x$ that $\lim f(x_n) = f(x)$. In particular, this holds for the sequence $x_n = x + 1/n$, which gives

$$\lim f_n(x) = \lim f(x + 1/n) = \lim f(x_n) = f(x).$$

(3 points)

Method 2. Let $x \in \mathbb{R}$ be fixed. Since f is continuous at x we have that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \epsilon.$$

Pick $n \in \mathbb{N}$ such that $1/n < \epsilon$ and set $\delta = \epsilon$. Then we have $|x - (x + 1/n)| < \delta$ which implies that

$$|f(x) - f_n(x)| = |f(x) - f(x + 1/n)| < \epsilon.$$

This proves that

$$\lim f_n(x) = f(x).$$

(3 points)

(b) Let $x \in \mathbb{R}$ be arbitrary. By the Mean Value Theorem there exists $x < c_n < x + 1/n$ such that

$$|f(x) - f_n(x)| = |\sin(x) - \sin(x + 1/n)| \leq |\cos(c_n)|\,|x - (x + 1/n)| \leq \frac{1}{n}$$

(3 points)

Method 1. This gives

$$\sup_{x \in \mathbb{R}} |f(x) - f_n(x)| \leq \frac{1}{n},$$

which implies that

$$\lim \left(\sup_{x \in \mathbb{R}} |f(x) - f_n(x)| \right) = 0.$$

We conclude that (f_n) converges uniformly to f on $\mathbb{R}$.
(3 points)

Method 2. Let $\epsilon > 0$ be arbitrary and pick $N \in \mathbb{N}$ such that $1/N < \epsilon$. If $n \geq N$, then

$$|f(x) - f_n(x)| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Since N does not depend on x we see that (f_n) converges uniformly to f on $\mathbb{R}$.
(3 points)

(c) Let $x \in \mathbb{R}$ be arbitrary. We have that

$$|f(x) - f_n(x)| = \left| 5x^2 - 5\left(x + \frac{1}{n}\right)^2 \right| = \left| \frac{10x}{n} + \frac{5}{n^2} \right|$$

(3 points)

Method 1. We have that

$$\sup_{x \in \mathbb{R}} |f(x) - f_n(x)| = \infty,$$

which immediately shows that (f_n) does *not* converge uniformly on $\mathbb{R}$.
(3 points)

Solution of problem 6 (3 + 6 + 3 + 3 = 15 points)

(a) For any partition $P = \{a = 0 < x_1 < x_2 < \cdots < x_n = \pi/2\}$ of the interval $[0, \pi/2]$ we have

$$m_k = \inf\{g(x) \,:\, x \in [x_{k-1}, x_k]\} = 0, \quad k = 1, \ldots, n.$$

This follows from the observation that any subinterval contains at least one irrational point at which g is zero. This gives the following lower sum:

$$L(g, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = 0.$$

(3 points)

(b) *Version 1.* For any subinterval we denote

$$M_k = \sup\{g(x) \,:\, x \in [x_{k-1}, x_k]\} \quad \text{and} \quad M_k' = \sup\{f(x) \,:\, x \in [x_{k-1}, x_k]\}$$

Since f is decreasing on the interval $[x_{k-1}, x_k]$ we have that $M_k' = \cos(x_{k-1})$.
(1 point)

Let (a_n) be any sequence of *rational* numbers in the interval $[x_{k-1}, x_k]$ such that $\lim a_n = x_{k-1}$. Then we have

$$\lim g(a_n) = \lim \cos(a_n) = \cos(x_{k-1}) = M_k',$$

where the second equality follows from the fact that the cosine is a continuous function. On the other hand, it is clear that $g(x) \leq M_k'$ for all $x \in [x_{k-1}, x_k]$. Therefore, we conclude that $M_k = M_k'$.
(3 points)

This gives the following upper sum:

$$U(g, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} M_k'(x_k - x_{k-1}) = U(f, P).$$

(2 points)

(c) Since f is integrable, we have that

$$\int_0^{\pi/2} f(x)\, dx \leq U(f, P).$$

By the Fundamental Theorem of Calculus we have

$$\int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} \cos(x)\, dx = [\sin(x)]_0^{\pi/2} = 1.$$

(3 points)

(d) For any partition P of $[0, \pi/2]$ we have shown that

$$U(g, P) - L(g, P) = U(f, P) \geq 1.$$

This show the function g is not integrable on $[0, \pi/2]$.
(3 points)